U9/b6-2

A
DISCOURSE
Concerning the
RESIDUAL ANALYSIS:
A NEW BRANCH of the
ALGEBRAIC ART,
Of very extenſive USE, both in *Pure Mathematics*
and *Natural Philoſophy*.

By JOHN LANDEN,
Inventor of the ſaid *Analyſis*, and Author of *Mathematical*
Lucubrations.

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Concerning the
RESIDUAL ANALYSIS.

BEFORE Sir ISAAC NEWTON invented the *Method of Fluxions*, mathematicians had made conſiderable improvements in the algebraic art; and had deviſed ſeveral curious rules for reſolving certain problems relating to the greateſt and leaſt ordinates, points of inflexion, tangents, curvature, and quadrature of curve lines; and the cubature of ſolids, *&c.* Thoſe rules, however, were eſteemed of little value, upon the appearance of the fluxionary method; which, being of far more extenſive uſe, was, by the mathematical world, received with great admiration, and ſtudied with great eagerneſs. Highly indeed has that method been extolled by many writers; yea, a certain gentleman has gone ſo far as to ſay, "The method of "fluxions is capable of reſolving ſuch difficulties as raiſe the "wonder and ſurprize of all mankind, and which would in

" vain be attempted by any other method whatſoever. So that " it is juſtly eſteemed the greateſt work of genius, and the " nobleſt thought that ever entered the human mind."

Pref. to EMERS. *Flux.*

Yet, notwithſtanding the method of fluxions is ſo greatly applauded, I am induced to think, it is not the moſt natural method of reſolving many problems to which it is uſually applied.—The operations therein being chiefly performed with algebraic quantities, it is, in fact, a branch of the algebraic art, or an improvement thereof, made by the help of ſome peculiar principles borrowed from the doctrine of motion: which principles, I muſt confeſs, to me ſeem not ſo properly applicable to algebra as thoſe on which that art was, before, very naturally founded. We may indeed very naturally conceive a line to be generated by motion; but there are quantities of various kinds, which we cannot conceive to be ſo generated. It is only in a figurative ſenſe, that an algebraic quantity can be ſaid to increaſe or decreaſe with ſome velocity or degree of ſwiftneſs; and, by the fluxion of a quantity of that kind, we muſt, I preſume, to have a clear idea of its meaning, underſtand, the velocity of a point ſuppoſed to deſcribe a line denoting ſuch quantity. Fluxions therefore are not immediately applicable to algebraic quantities; but in fluxionary computations made by means of ſuch quantities, we, to proceed with perſpicuity, muſt have recourſe to the ſuppoſition of lines being put to denote thoſe quantities, and the generation of thoſe lines by motion. It therefore, to me, ſeems more proper, in the inveſtigation of propoſitions by algebra, to proceed upon the *anciently-received* principles of that art, than to introduce therein, without any neceſſity, the new fluxionary principles, derived from a conſideration of motion; and the rather, as the introduction of thoſe new principles is not attended with any peculiar advantage.—That the borrowing principles from the doctrine of motion, with a view to improve the analytic art, was done, not only without any neceſſity, but even without any peculiar advantage, will appear by ſhewing, that whatever can be done by the method of computation, which is founded on thoſe borrowed principles, may be done, as well, by another method founded entirely on the *anciently-received* principles of algebra: And that I ſhall endeavour to ſhew, as ſoon as I have leiſure, in the treatiſe I lately propoſed to publiſh by ſubſcription.—In the mean time, this eſſay is intended to give the inquiſitive reader ſome notion of the new method of computation, which is the ſubject of that treatiſe.—Which method I call the *Reſidual Analyſis*; becauſe, in all the enquiries wherein it is made uſe of, the concluſions are obtained by means of reſidual quantities.

In the application of the *Reſidual Analyſis*, a geometrical or phyſical problem is naturally reduced to another purely algebraical; and the ſolution is then readily obtained, without any ſuppoſition of motion, and without conſidering quantities as compoſed of infinitely ſmall particles.

It is by means of the following theorem, *viz.*

$$\frac{x^{\frac{m}{n}} - v^{\frac{m}{n}}}{x - v} = x^{\frac{m}{n}-1} \times \frac{1 + \frac{v}{x} + \overline{\frac{v}{x}}\Big|^{2} + \overline{\frac{v}{x}}\Big|^{3} \qquad (m)}{1 + \overline{\frac{v}{x}}\Big|^{\frac{m}{n}} + \overline{\frac{v}{x}}\Big|^{\frac{2m}{n}} + \overline{\frac{v}{x}}\Big|^{\frac{3m}{n}} \qquad (n)},$$

(where *m* and *n* are any integers,)

that we are enabled to perform all the principal operations in our ſaid Analyſis; and I am not a little ſurprized, that a theorem ſo obvious, and of ſuch vaſt uſe, ſhould ſo long eſcape the notice of algebraiſts!

I have no objection againſt the truth of the method of fluxions, being fully ſatisfied, that even a problem purely algebraical may be very clearly reſolved by that method, by bringing into conſideration lines, and their generation by motion. But I muſt own, I am inclined to think, ſuch a problem would be more naturally reſolved by pure algebra, without any ſuch conſideration of lines and motion. — Suppoſe it required to inveſtigate the binomial theorem; *i. e.* to expand $\overline{1+x}\Big|^{\frac{m}{n}}$ into a ſeries of terms of x, and known coefficients.

To do this by the method of fluxions, we firſt aſſume

$$\overline{1+x}^{\frac{m}{n}} = 1 + ax + bx^2 + cx^3 + dx^4 \text{ \&c.}$$

We, to proceed with perſpicuity, are next to conceive x, and each term of that aſſumed equation, to be denoted by ſome line, and that line to be deſcribed by the motion of a point: Then, ſuppoſing $\dot{x}$ to be the velocity of the point deſcribing the line x, and taking, by the rules taught by thoſe who have treated of the ſaid method, the ſeveral contemporary velocities of the other deſcribing points, or the fluxions of the ſeveral terms in the ſaid equation, we get

$$\frac{m}{n} \times \overline{1+x}^{\frac{m}{n}-1} \times \dot{x} = a\dot{x} + 2bx\dot{x} + 3cx^2\dot{x} + 4dx^3\dot{x} \text{ \&c.}$$

becauſe, when the ſpace deſcribed by a motion is always equal to the ſum of the ſpaces deſcribed in the ſame time by any other motions, the velocity of the firſt motion is always equal to the ſum of the velocities of the other motions.

From which laſt equation, by dividing by $\dot{x}$, or ſuppoſing $\dot{x}$ equal to unity, we have

$$\frac{m}{n} \times \overline{1+x}^{\frac{m}{n}-1} = a + 2bx + 3cx^2 + 4dx^3 \text{ \&c.}$$

Conſequently, multiplying by $1+x$, we have

$$\frac{m}{n} \times \overline{1+x}^{\frac{m}{n}}, \text{ or its equal } \frac{m}{n} + \frac{m}{n}ax + \frac{m}{n}bx^2 + \frac{m}{n}cx^3 \text{ \&c.}$$

$$= a + \left.\begin{matrix}2b\\a\end{matrix}\right\} x + \left.\begin{matrix}3c\\2b\end{matrix}\right\} x^2 + \left.\begin{matrix}4d\\3c\end{matrix}\right\} x^3 \text{ \&c.}$$

From whence, by comparing the homologous terms, the coefficients a, b, c, &c. will be found.

The ſame theorem is inveſtigated by the *Reſidual Analyſis*, in the following manner.

Aſſuming, as above,

$$\overline{1+x}^{\frac{m}{n}} = 1 + ax + bx^2 + cx^3 \text{ \&c.}$$

we have $\overline{1+y}^{\frac{m}{n}} = 1 + ay + by^2 + cy^3$ &c.
and, by ſubtraction,

$$\overline{1+x}^{\frac{m}{n}} - \overline{1+y}^{\frac{m}{n}} = a\,.\,\overline{x-y} + b\,.\,\overline{x^2-y^2} + c\,.\,\overline{x^3-y^3} + d\,.\,\overline{x^4-y^4}$$
&c.

If, now, we divide by the reſidual $x-y$, we ſhall get

$$\overline{1+x}^{\frac{m}{n}-1} \times \frac{1 + \frac{1+y}{1+x} + \left.\overline{\frac{1+y}{1+x}}\right|^{2} + \left.\overline{\frac{1+y}{1+x}}\right|^{3} \quad (m)}{1 + \left.\overline{\frac{1+y}{1+x}}\right|^{\frac{m}{n}} + \left.\overline{\frac{1+y}{1+x}}\right|^{\frac{2m}{n}} + \left.\overline{\frac{1+y}{1+x}}\right|^{\frac{3m}{n}} \quad (n)}$$ [*]

$$= a + b\,.\,\overline{x+y} + c\,.\,\overline{x^2+xy+y^2} + d\,.\,\overline{x^3+x^2y+xy^2+y^3} \text{ \&c.}$$

which equation muſt hold true let y be what it will: From whence, by taking y equal to x, we find, as before,

$$\frac{m}{n} \times \overline{1+x}^{\frac{m}{n}-1} = a + 2bx + 3cx^2 + 4dx^3 \text{ \&c.}$$

The reſt of the operation will therefore be as above ſpecified.

Now, as to either of theſe methods of inveſtigation, I ſhall not take upon me to ſay any thing in particular; it is ſubmitted to the reader to compare one with the other, and judge which of the two is moſt natural.

An algebraic fractional quantity being ſo compoſed of x and known quantities, that the numerator and denominator both vaniſh when x is therein taken equal to ſome certain quantity (r); and it being required to find the value of ſuch fraction when x is ſo taken: a fluxioniſt will, from the conſideration of the deſcription of certain lines by motion, perceive, that, taking the fluxion of the numerator, and likewiſe that of the denominator, (x being ſuppoſed variable,) the quotient ariſing by dividing the former fluxion by the latter, will, when x is therein taken equal to r, be the value required to be found. But will it not, in reſolving ſuch a problem, be more natural and ſatisfactory to give the general value of ſuch fractional quantity in terms which ſhall not vaniſh when x is equal to r; from whence the value of the ſaid quantity may be known as well in the particular caſe, when x is equal to r, as in any other caſe? or, from a conſideration of the manner of obtaining ſuch general value, to deduce (as we eaſily may) a brief rule for aſſigning ſuch particular value only?—If it be aſked, "How ſhall we aſſign ſuch general value, without the vaſt "trouble of taking away radical ſigns?" I anſwer, "by the "*Reſidual Analyſis*;" and for the reader's ſatisfaction ſhall here give an Example.

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[*] See note 1. at the end.

Let it be propoſed to expreſs the general value of the fraction $\frac{\sqrt{2r^3x-x^4}-\sqrt{rx^3}}{r-\sqrt{rx}}$ (whoſe numerator and denominator both vaniſh when x is equal to r) in terms which ſhall not vaniſh when x is equal to r.

Seeing the expreſſions $\sqrt{2r^3x-x^4}-\sqrt{rx^3}$ and $r-\sqrt{rx}$ both vaniſh when x is equal to r; it is obvious, that, if, in the quotient of $\overline{\sqrt{2r^3x-x^4}-\sqrt{rx^3}}-\overline{\sqrt{2r^3y-y^4}-\sqrt{ry^3}}$ divided by $\overline{r-\sqrt{rx}}-\overline{r-\sqrt{ry}}$, y be taken equal to r, the reſulting expreſſion will be equal to $\frac{\sqrt{2r^3x-x^4}-\sqrt{rx^3}}{r-\sqrt{rx}}$, let x be what it will.

Now $\frac{\overline{\sqrt{2r^3x-x^4}-\sqrt{rx^3}}-\overline{\sqrt{2r^3y-y^4}-\sqrt{ry^3}}}{\overline{r-\sqrt{rx}}-\overline{r-\sqrt{ry}}}$ is manifeſtly $=-\frac{\sqrt{2r^3x-x^4}-\sqrt{2r^3y-y^4}}{r^{\frac{1}{2}}\times x^{\frac{1}{2}}-y^{\frac{1}{2}}}+\frac{x^{\frac{3}{2}}-y^{\frac{3}{2}}}{x^{\frac{1}{2}}-y^{\frac{1}{2}}}$, which being $=\frac{x^3+x^2y+xy^2+y^3-2r^3}{\sqrt{2r^3x-x^4}+\sqrt{2r^3y-y^4}}\div\frac{r^{\frac{1}{2}}}{x^{\frac{1}{2}}+y^{\frac{1}{2}}}+x+x^{\frac{1}{2}}y^{\frac{1}{2}}+y$,* we have, by taking y equal to r,

$$\frac{\sqrt{2r^3x-x^4}-\sqrt{rx^3}}{r-\sqrt{rx}}=\frac{x^3+rx^2+r^2x-r^3}{\sqrt{2r^3x-x^4}+r^2}\div\frac{r^{\frac{1}{2}}}{x^{\frac{1}{2}}+r^{\frac{1}{2}}}+$$

$x+r^{\frac{1}{2}}x^{\frac{1}{2}}+r$, the expreſſion ſought.

Whence it is evident, that, when x is equal to r,

$$\frac{\sqrt{2r^3x-x^4}-\sqrt{rx^3}}{r-\sqrt{rx}} \text{ is } = 5r.$$

To find the value of ſuch a fraction, in the particular caſe when its numerator and denominator both vaniſh together, was, ſometime ago, a famous problem; which, for a long time, (as we are told by the propoſer, Mr. JOHN BERNOULLI,) exerciſed the ſkill of the *celebrated Marquis* DE L'HOSPITAL, and other eminent mathematicians, who attempted to reſolve it, but without ſucceſs. At length Mr. BERNOULLI communicated his method of ſolution, which in fact is the ſame as our firſt-mentioned method, derived from the doctrine of fluxions. Vid. *Acta Erud. Lipſ.* 1704.

Fluxioniſts, in determining the limit of the ratio of the increments of x and $x^{\frac{m}{n}}$, commonly have recourſe to the binomial theorem, (which is much more difficult to inveſtigate than the limit they are ſeeking): But how eaſily may that limit be found, without the help of that theorem, by the equation exhibited in page 5! Thus, the increment of x being denoted by x', the increment of $x^{\frac{m}{n}}$ is $\overline{x+x'}^{\frac{m}{n}}-x^{\frac{m}{n}}$, and the ratio of thoſe increments is

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* See notes 2. 3. 4. at the end.

$$\frac{\overline{x+x'}^{\frac{m}{n}}-x^{\frac{m}{n}}}{x'}=\frac{\overline{x+x'}^{\frac{m}{n}}-x^{\frac{m}{n}}}{\overline{x+x'}-x}=$$

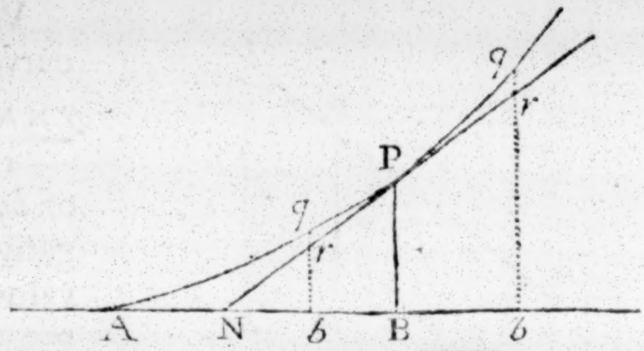
$$\overline{x+x'}^{\frac{m}{n}-1}\times\frac{1+\frac{x}{x+x'}+\overline{\frac{x}{x+x'}}\Big|^{2}+\overline{\frac{x}{x+x'}}\Big|^{3}\quad(m)}{1+\overline{\frac{x}{x+x'}}\Big|^{\frac{m}{n}}+\overline{\frac{x}{x+x'}}\Big|^{\frac{2m}{n}}+\overline{\frac{x}{x+x'}}\Big|^{\frac{3m}{n}}\quad(n)},$$

which, when x' vaniſhes, is manifeſtly equal to $\frac{m}{n}x^{\frac{m}{n}-1}$, the limit of the ſaid ratio.

The method of fluxions is never more elegantly applied, than in drawing tangents to curve lines.—In performing that buſineſs by that method, we conſider the deſcription of the curve by motion; and finding, that the ſubtangent bears the ſame proportion to the ordinate, as the velocity of the deſcribing point in the direction of the baſe to its velocity in the direction of the ordinate; we, by means of the equation of the curve, and the rules for finding the fluxions of quantities, readily compute the ſubtangent's length.—The ſame buſineſs is performed (how elegantly I ſhall not preſume to determine) by the *Reſidual Analyſis,* after the following manner. I conſider the curve as already deſcribed, without any regard to its generation, and find the value of a certain line, (terminated by the curve and its tangent,) in algebraic terms involving *(s)* the ſubtangent with other quantities; which algebraic expreſſion I obſerve, from an obvious property of the line it is found to denote, muſt have a certain property with reſpect to being poſitive or negative in certain caſes. I therefore aſſume that expreſſion equal to another which is known to have that very property; and from thence, by means of the theorem mentioned in page 5. readily find the required value of *s*.—This the following proceſs will more clearly explain.

If

Fig. 1.



If N*r*P*r* touch the curve A*q*P*q* in P; and, *brq* being parallel to BP, if AB be called x; BP, y; A*b*, v; *bq*, u; and the ſubtangent NB, s: then will N*b* be $= s - x \pm v$, $br = y - \frac{y \times \overline{x-v}}{s}$, and the reſidual $bq - br\ (= qr) = \frac{y \times \overline{x-v}}{s} - \overline{y-u}$.

Now, *brq* being drawn on either ſide of BP, *bq* is manifeſtly greater than *br*: therefore $\frac{y \times \overline{x-v}}{s} - \overline{y-u}$ (the value of $bq - br$) muſt be always poſitive, when, x being of any value whatever, v is either leſs or greater than x: it muſt likewiſe be obſerved, that the value of $bq - br$ vaniſhes when v is equal to x.

We are now to aſſume another expreſſion (compoſed of v and other quantities) that ſhall vaniſh when v is equal to x, and be always poſitive when, x being of any value whatever, v is either leſs or greater than x:—ſuch is the following expreſſion. viz. $\overline{x-v}^{\,m} \times Q$; m being an even poſitive number, or a poſitive fraction whoſe numerator is an even number, and denominator an odd number; and Q an algebraic expreſſion ſo compoſed of v and other quantities, that (q) its value, when v is equal to x, is finite and poſitive *. It is obvious therefore, that (ſuppoſing m and Q to be as juſt now ſpecified) if

$$\frac{y \times \overline{x-v}}{s} - \overline{y-u} \text{ be aſſumed } = \overline{x-v}^{\,m} \times Q,$$

ſuch aſſumed equation muſt hold true let v be what it will, and the value of s from thence determined will be the ſubtangent correſponding to the tangent at the point P.

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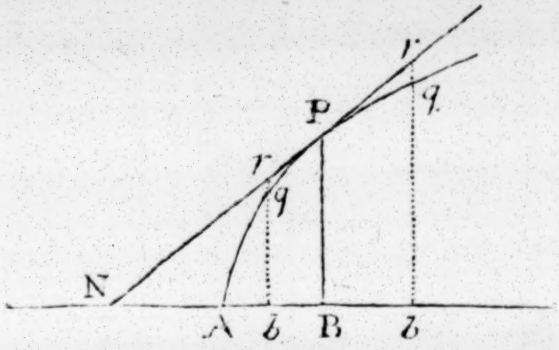
* See note 5. at the end.

Fig. 2.



When the convexity of the curve is upwards, as in Fig. 2. $\frac{y\times\overline{x-v}}{s} - \overline{y-u}$ (the value of $bq-br$) will be always negative, when, x being of any value whatever, v is either greater or leſs than x; and our aſſumed equation muſt be the ſame as above, except that q muſt be negative inſtead of poſitive.

To find s, let each ſide of the aſſumed equation be divided by $x-v$; by which means we get

$$\frac{y}{s} - [x\,|\,y] = \overline{x-v}^{\,m-1} \times Q,$$

$[x\,|\,y]$ being put for the quotient of $y-u$ divided by $x-v$. Now, when v is equal to x, the expreſſion $\overline{x-v}^{\,m-1} \times Q$ or its reciprocal will vaniſh, according as m is greater or leſs than 1. ——By ſuppoſing ſuch reciprocal to vaniſh, we have in general $s=0$, which is abſurd: therefore m muſt be greater than 1; and, conſequently, by taking v equal to x, and writing $[x\perp y]$ for the value of $[x\,|\,y]$ in the particular caſe when v is ſo taken, we have $\frac{y}{s} - [x\perp y] = 0$, and $s = \frac{y}{[x\perp y]}$.

If, now, this value of s be ſubſtituted above, we ſhall have $[x\perp y] - [x\,|\,y] = \overline{x-v}^{\,m-1} \times Q$; in which equation, it is eaſy to prove, (but I ſhall not ſtay to do it here,) that, q and its reciprocal being finite, m will in general be equal to 2: which being agreeable to our ſuppoſition, it follows, that s is rightly determined; and q muſt of neceſſity be poſitive or negative, according as the convexity of the curve is downwards or upwards.

Therefore, if, from the aſſumed equation and the equation of the curve, the value of q be computed, we may from thence, without farther enquiry, know whether the curve, at any point thereof, be convex or concave towards the baſe.—— Now, by what is ſaid above, we have

$$[x \perp y] - [x \mid y] \ \left(= \overline{x - v}^{\,m-1} \times Q\right) = \overline{x - v} \times Q;$$

from whence we have $Q = \dfrac{[x \perp y] - [x \mid y]}{x - v}$.

Conſequently q will be equal to the value of the quotient of $[x \perp y] - [x \mid y]$ divided by $x - v$, in the particular caſe when v is equal to x. Now, putting $[v \perp u]$ for the quantity which reſults by writing v and u inſtead of x and y reſpectively, in the expreſſion denoted by $[x \perp y]$; it is eaſy to prove, by the method pointed out in page 8. that ſuch particular value of the ſaid quotient will be equal to *half* the quotient of $[x \perp y] - [v \perp u]$ divided by $x - v$, when v is therein taken equal to x. Which laſt-mentioned quotient will be more eaſily computed than the quotient of $[x \perp y] - [x \mid y]$ divided by $x - v$.

EXAMPLE I. *Let it be propoſed to draw a tangent to the parabola, whoſe equation is* $ax = y^2$.

ax being $= y^2$, $a^{\frac{1}{2}} x^{\frac{1}{2}}$ will be $= y$, $a^{\frac{1}{2}} v^{\frac{1}{2}} = u$, and $a^{\frac{1}{2}} x^{\frac{1}{2}} - a^{\frac{1}{2}} v^{\frac{1}{2}} = y - u$; from whence, by dividing by $x - v$, we have $\dfrac{a^{\frac{1}{2}}}{x^{\frac{1}{2}} + v^{\frac{1}{2}}} = [x \mid y]$; and hence, by taking v equal to x, $[x \perp y] = \dfrac{a^{\frac{1}{2}}}{2x^{\frac{1}{2}}}$.

Conſequently, ſubſtituting $a^{\frac{1}{2}} x^{\frac{1}{2}}$ and $\dfrac{a^{\frac{1}{2}}}{2x^{\frac{1}{2}}}$ inſtead of y and $[x \perp y]$ reſpectively in the equation $s = \dfrac{y}{[x \perp y]}$, it appears that the ſubtangent s is $= 2x$.

EXAMPLE II. *The equation of the curve being* $ax^{\frac{m}{n}}=y$, $av^{\frac{m}{n}}$ will be $=u$, and $ax^{\frac{m}{n}}-av^{\frac{m}{n}}=y-u$; from whence, by dividing by $x-v$, we get

$$a\times x^{\frac{m}{n}-1}\times\frac{1+\frac{v}{x}+\left.\overline{\frac{v}{x}}\right|^{2}+\left.\overline{\frac{v}{x}}\right|^{3}\ (m)}{1+\left.\overline{\frac{v}{x}}\right|^{\frac{m}{n}}+\left.\overline{\frac{v}{x}}\right|^{\frac{2m}{n}}+\left.\overline{\frac{v}{x}}\right|^{\frac{3m}{n}}\ (n)}=[x\mid y]\text{; and hence,}$$

by taking v equal to x, we have $[x\perp y]=\frac{am}{n}\times x^{\frac{m}{n}-1}$.

Therefore, by writing $ax^{\frac{m}{n}}$ and $\frac{am}{n}x^{\frac{m}{n}-1}$ inſtead of y and $[x\perp y]$ reſpectively in the equation $s=\frac{y}{[x\perp y]}$, we find $s=\frac{n}{m}x$.

Moreover, ſince $[x\perp y]$ is $=\frac{am}{n}x^{\frac{m}{n}-1}$, $[v\perp u]$ will be $=\frac{am}{n}v^{\frac{m}{n}-1}$; and the quotient of $[x\perp y]-[v\perp u]$ divided by $x-v$ will be equal

$$\text{to }\frac{am}{n}\times x^{\frac{m}{n}-2}\times\frac{1+\frac{v}{x}+\left.\overline{\frac{v}{x}}\right|^{2}+\left.\overline{\frac{v}{x}}\right|^{3}\ (m-n)}{1+\left.\overline{\frac{v}{x}}\right|^{\frac{m-n}{n}}+\left.\overline{\frac{v}{x}}\right|^{\frac{2m-2n}{n}}+\left.\overline{\frac{v}{x}}\right|^{\frac{3m-3n}{n}}\ (n)}\text{, which, when}$$

v is $=x$, is equal to $\frac{am\times\overline{m-n}}{n^{2}}\times x^{\frac{m}{n}-2}$.

Therefore q will be $=\frac{am\times\overline{m-n}}{2n^{2}}\times x^{\frac{m}{n}-2}$; and, at the point correſponding to x, the convexity of the curve will be downwards or upwards, according as that value of q is poſitive or negative.

EXAMPLE III. *The equation of the curve being* $ay^2 - xy^2 = x^3$, $au^2 - vu^2$ will be $= v^3$, and $ay^2 - xy^2 - \overline{au^2 - vu^2}$ $(= a \times \overline{y^2 - u^2} - \overline{xy^2 - vu^2}) = x^3 - v^3$; from whence, by dividing by $x - v$, we have $a \times \overline{y + u} \times [x|y] - y^2 - v \times \overline{y + u} \times [x|y] = x^2 + xv + v^2$; and from hence we get $[x|y] = \frac{x^2 + xv + v^2 + y^2}{\overline{a - v} \times \overline{y + u}}$. Therefore, taking v and u equal to x and y reſpectively, we have $[x \perp y] = \frac{3x^2 + y^2}{2y \times \overline{a - x}}$: Which laſt expreſſion being ſubſtituted for $[x \perp y]$ in the equation $s = \frac{y}{[x \perp y]}$, we find $s = \frac{2y^2 \times \overline{a - x}}{3x^2 + y^2}$. Moreover, ſince $[x \perp y]$ is $= \frac{3x^2 + y^2}{2y \times \overline{a - x}}$, $[v \perp u]$ will be $= \frac{3v^2 + u^2}{2u \times \overline{a - v}}$; and the quotient of $[x \perp y] - [v \perp u]$ divided by $x - v$ will be equal to $\frac{3x + 3v + \overline{y + u} \times [x|y]}{2y \times \overline{a - x}} - \overline{3v^2 + u^2} \times \frac{\overline{a - x} \times [x|y] - y}{2uy \times \overline{a - v} \times \overline{a - x}}$, which, when v is $= x$, is equal to $\frac{6x + 2y[x \perp y]}{2y \times \overline{a - x}} - \overline{3x^2 + y^2} \times \frac{\overline{a - x} \times [x \perp y] - y}{2y^2 \times \overline{a - x}^2}$ $= \frac{\overline{3ax + 3x^2 + 3y^2} \times y^3}{4x^6}$, as appears by ſubſtitution.

Therefore q will be $= \frac{\overline{3ax + 3x^2 + 3y^2} \times y^3}{8x^6}$; which, being poſitive or negative according as y is poſitive or negative, indicates that the branch of the curve above the baſe has its convexity always downwards, and the branch below the baſe its convexity always upwards; and conſequently, that both branches are always convex towards the baſe.

Having ſhewn how our Analyſis is applied in drawing tangents to curves, let us conſider its application in reſolving problems relating to the *maxima* and *minima* of quantities.

Fig. 3.
4.

Fig. 3.

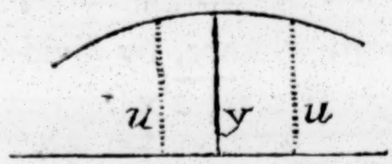
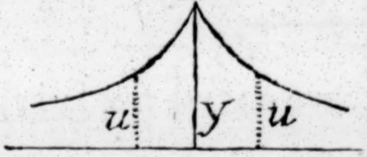


Fig. 4.



Suppoſe y to be the ordinate of a curve correſponding to the abſciſſa x; and u another ordinate (of the ſame curve) correſponding to the abſciſſa v. Then if y be a maximum it will be greater than u, whether v be leſs or greater than x; and conſequently the reſidual $y-u$ will, in ſuch caſe, be always poſitive when v is either leſs or greater than x, (between certain limits,) and will vaniſh when v is equal to x.——Now, as we have obſerved in page 11. the expreſſion $\overline{x-v}^{\,m} \times Q$ vaniſhes when v is equal to x, and is always poſitive when v is either leſs or greater than x; m being an even poſitive number, or a poſitive fraction whoſe numerator is an even number, and denominator an odd number; and Q an algebraic expreſſion ſo compoſed of v and other quantities, that (q) its value, when v is equal to x, is finite and poſitive.

It therefore evidently follows, that, (ſuppoſing m and Q to be as here ſpecified,) if

$$y-u \text{ be aſſumed } = \overline{x-v}^{\,m} \times Q,$$

the value of x from thence determined will be the value of the abſciſſa, whoſe correſpondent ordinate is a maximum.

Fig. 5.
6.

Fig. 5.

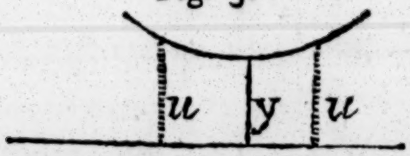
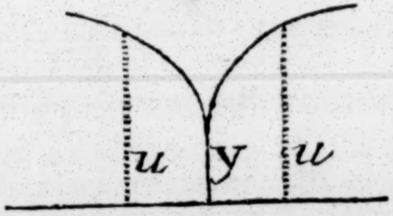


Fig. 6.



If y be a minimum, $y-u$ will be always negative when v is either leſs or greater than x, (between certain limits;) and our aſſumed equation muſt be the ſame as above, except that q muſt be negative inſtead of poſitive.

To find the value, or values, which x may have in the aſſumed equation, let each ſide thereof be divided by $x-v$; by which means we get $[x \perp y] = \overline{x-v}^{\,m-1} \times Q$; and hence, by taking v equal to x, we have $[x \perp y] = 0$, or $\frac{1}{[x \perp y]} = 0$; $\overline{x-v}^{\,m-1} \times Q$ or its reciprocal then vaniſhing, according as m is greater or leſs than unity.

From what is ſaid it is evident, that, x being as determined by either of the two equations juſt now deduced from the aſſumed one, y is a maximum or a minimum, according as q is poſitive or negative, m being as above ſpecified: but, q and its reciprocal being finite, if m comes out contrary to our ſuppoſition, the abſciſſa ſo determined will not correſpond to an ordinate that is either a maximum or a minimum.—This we ſhall, by and by, explain by proper examples.

Dividing each ſide of the aſſumed equation by $\overline{x-v}^{\,m}$, we have $\frac{y-u}{\overline{x-v}^{\,m}} = Q$. Now, by the method pointed out in page 8. it is eaſy to prove, that, ſuppoſing x equal to ſome certain quantity b, the value of the quotient of $y-u$ divided by $\overline{x-v}^{\,m}$, in the particular caſe when v is alſo equal to b, is equal to the quotient of $[x \perp y]$ divided by $m \times \overline{b-x}^{\,m-1}$, when x is therein taken equal to b. Conſequently, ſuppoſing b to denote a value of x in the equation $[x \perp y] = 0$, or $\frac{1}{[x \perp y]} = 0$; q will be equal to the value of $\frac{[x \perp y]}{m \times \overline{b-x}^{\,m-1}}$, when x is equal to b. It appears therefore, that, by reſolving $[x \perp y]$ into two ſuch factors, (F and G,) that one of them (F) ſhall be ſome power of $b-x$, and the other (G) ſhall neither vaniſh nor become infinite when x is therein taken equal to b; we may find m and q, and ſo obtain a clear ſolution to any problem *de maximis et minimis*, without aſſigning the general value of Q: for by comparing $\overline{b-x}^{\,m-1}$ with F, m will be known; and q will be the value of $\frac{G}{m}$ when x is equal to b.

EXAMPLE I. *Suppoſe* $y=ax-x^2$, *where* a *is invariable.*

Then we ſhall have $[x \perp y] = a-2x = \overline{\frac{a}{2}-x} \times 2 = 0$, where x is $= \frac{a}{2}$.

Now, b being $= \frac{a}{2}$, we have $F = \frac{a}{2} - x = \overline{b-x}^{\,m-1}$, and $G=2$. Therefore, $m-1$ being manifeſtly $= 1$, m is $= 2$; and, q (the value of $\frac{G}{m}$ when x is $= b$) being a poſitive quantity, y is a maximum when x is $= \frac{a}{2}$.

EXAMPLE II. *Suppoſe* $y=x^4-a^3x$, *where* a *is invariable.* Then we have

$[x \perp y] = 4x^3 - a^3 = \overline{\frac{a}{4^{\frac{1}{3}}} - x} \times \overline{-4^{\frac{1}{3}}a^2 - 4^{\frac{2}{3}}ax - 4x^2} = 0$, where the real value of x is $\frac{a}{4^{\frac{1}{3}}}$.

Now, b being $= \frac{a}{4^{\frac{1}{3}}}$, we have $F = \frac{a}{4^{\frac{1}{3}}} - x = \overline{b-x}^{\,m-1}$, and $G = -4^{\frac{1}{3}}a^2 - 4^{\frac{2}{3}}ax - 4x^2$. Therefore, $m-1$ being evidently $= 1$, m is $= 2$; and, q (the value of $\frac{G}{m}$ when x is $= b$) being a negative quantity, y is a minimum when x is $=\frac{a}{4^{\frac{1}{3}}}$.

EXAMPLE III. *Suppoſe* $y=a^4x-a^3x^2-\frac{2}{3}a^2x^3+\frac{3}{2}ax^4-\frac{3}{5}x^5$, *where* a *is an invariable poſitive quantity.*

Then we ſhall have

$$[x\perp y]=a^4-2a^3x-2a^2x^2+6ax^3-3x^4=$$

$$3\times\overline{\frac{a}{\sqrt{3}}-x}\times\overline{\frac{a}{\sqrt{3}}+x}\times\overline{a-x}^{2}=0;$$

where x is equal to $\frac{a}{\sqrt{3}}$, or $-\frac{a}{\sqrt{3}}$, or a.

Taking b equal to $\frac{a}{\sqrt{3}}$, we have $F=\frac{a}{\sqrt{3}}-x=\overline{b-x}^{m-1}$, and $G=3\times\overline{\frac{a}{\sqrt{3}}+x}\times\overline{a-x}^{2}$. Therefore, $m-1$ being $=1$, m is $=2$; and, q (the value of $\frac{G}{m}$ when x is $=b$) being a poſitive quantity, y is a maximum when x is $=\frac{a}{\sqrt{3}}$.

Taking b equal to $-\frac{a}{\sqrt{3}}$, we have $F=-\frac{a}{\sqrt{3}}-x=\overline{b-x}^{m-1}$, and $G=3\times\overline{x-\frac{a}{\sqrt{3}}}\times\overline{a-x}^{2}$. Therefore, $m-1$ being $=1$, m is $=2$; and, q (the value of $\frac{G}{m}$ when x is $=b$) being a negative quantity, y is a minimum when x is $=-\frac{a}{\sqrt{3}}$.

Taking b equal to a, we have $F=\overline{a-x}^{2}=\overline{b-x}^{m-1}$, and $G=3\times\overline{\frac{a}{\sqrt{3}}-x}\times\overline{\frac{a}{\sqrt{3}}+x}$. Therefore, $m-1$ being manifeſtly $=2$, m is $=3$; which being contrary to our ſuppoſition, it follows, that y is neither a maximum nor a minimum when x is $= a$, notwithſtanding this value of x is determined in the ſame manner as are the other two values above ſpecified.

EXAMPLE IV. *Suppoſe* $y = x + \overline{a^3 - x^3}^{\frac{1}{3}}$, *where* a *is an invariable poſitive quantity.* ✓

Then we ſhall have

$$[x \perp y] = \frac{\overline{a^3 - x^3}^{\frac{2}{3}} - x^2}{\overline{a^3 - x^3}^{\frac{2}{3}}} \left(= \frac{a^6 - 2a^3x^3}{\overline{a^3 - x^3}^{\frac{2}{3}} \times \overline{n^2 + 3nx^2 + 3x^4}} {}^{*} = \frac{\overline{\frac{a}{2^{\frac{1}{3}}} - x} \times \overline{2^{\frac{1}{3}}a^5 + 2^{\frac{2}{3}}a^4x + 2a^3x^2}}{\overline{a^3 - x^3}^{\frac{2}{3}} \times \overline{n^2 + 3nx^2 + 3x^4}}\right.$$

, n being put for $\overline{a^3 - x^3}^{\frac{2}{3}} - x^2$)

$$= 0, \text{ and } \frac{1}{[x \perp y]} = \frac{\overline{a^3 - x^3}^{\frac{2}{3}}}{\overline{a^3 - x^3}^{\frac{2}{3}} - x^2} = \frac{\overline{a - x}^{\frac{2}{3}} \times \overline{a^2 + ax + x^2}^{\frac{2}{3}}}{\overline{a^3 - x^3}^{\frac{2}{3}} - x^2} = 0.$$

In the equation $[x \perp y] = 0$, the real value of x is $\frac{a}{2^{\frac{1}{3}}}$; and, in the equation $\frac{1}{[x \perp y]} = 0$, the real value of x is a.

Taking b equal to $\frac{a}{2^{\frac{1}{3}}}$, we have $F = \frac{a}{2^{\frac{1}{3}}} - x = \overline{b - x}^{m-1}$, and

$$G = \frac{2^{\frac{1}{3}}a^5 + 2^{\frac{2}{3}}a^4x + 2a^3x^2}{\overline{a^3 - x^3}^{\frac{2}{3}} \times \overline{n^2 + 3nx^2 + 3x^4}}.$$

Therefore, $m - 1$ being $= 1$, m is $= 2$; and, q (the value of $\frac{G}{m}$ when x is $= b$) being a poſitive quantity, y is a maximum when x is $= \frac{a}{2^{\frac{1}{3}}}$.

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* See note 6. at the end.

Taking b equal to a, we have $F = \overline{a-x}^{\,-\frac{2}{3}} = \overline{b-x}^{\,m-1}$,

and $G = \dfrac{\overline{a^3-x^3}^{\frac{2}{3}} - x^2}{\overline{a^2+ax+x^2}^{\frac{2}{3}}}$. Therefore, $m-1$ being evidently $= -\frac{2}{3}$, m is $= \frac{1}{3}$; which being contrary to our ſuppoſition, it follows, that y is neither a maximum nor a minimum when x is $= a$.

EXAMPLE V. *Suppoſe* $y = x^2 + \overline{a^3 - x^3}^{\frac{2}{3}}$, *where* a *is an invariable poſitive quantity.*

Then we ſhall have

$$[x \perp y] = \frac{2x \times \overline{a^3-x^3}^{\frac{1}{3}} - 2x^2}{\overline{a^3-x^3}^{\frac{1}{3}}} \left(= \frac{8a^3x - 16x^4}{\overline{a^3-x^3}^{\frac{1}{3}} \times \overline{g^2x^{-2} + 6g + 12x^2}} = \right.$$

$$\frac{-8 \times \overline{0-x} \times \overline{\frac{a}{2^{\frac{1}{3}}} - x} \times \overline{2^{\frac{1}{3}}a^2 + 2^{\frac{2}{3}}ax + 2x^2}}{\overline{a^3-x^3}^{\frac{1}{3}} \times \overline{g^2x^{-2} + 6g + 12x^2}},\ g \text{ being put for}$$

$$\left. 2x \times \overline{a^3-x^3}^{\frac{1}{3}} - 2x^2\right) = 0, \text{ and } \frac{1}{[x \perp y]} = \frac{\overline{a^3-x^3}^{\frac{1}{3}}}{2x \times \overline{a^3-x^3}^{\frac{1}{3}} - 2x^2} =$$

$$\frac{\overline{a-x}^{\frac{1}{3}} \times \overline{a^2+ax+x^2}^{\frac{1}{3}}}{2x \times \overline{a^3-x^3}^{\frac{1}{3}} - 2x^2} = 0.$$

In the equation $[x \perp y] = 0$, the real values of x are 0, and $\frac{a}{2^{\frac{1}{3}}}$; and, in the equation, $\frac{1}{[x \perp y]} = 0$, the real value of x is a.

Taking b equal to o, we have $F = 0 - x = \overline{b - x}^{\,m-1}$, and $G = \dfrac{8 \times \overline{x - \frac{a}{2^{\frac{1}{3}}}} \times \overline{2^{\frac{1}{3}}a^2 + 2^{\frac{2}{3}}ax + 2x^2}}{\overline{a^3 - x^3}^{\,\frac{1}{3}} \times \overline{g^2x^{-2} + 6g + 12x^2}}$. Therefore, $m - 1$ being $= 1$, m is $= 2$; and, q (the value of $\frac{G}{m}$ when x is $= b$) being a negative quantity, y is a minimum when x is $= 0$.

Taking b equal to $\frac{a}{2^{\frac{1}{3}}}$, we have $F = \frac{a}{2^{\frac{1}{3}}} - x = \overline{b - x}^{\,m-1}$, and $G = \dfrac{8x \times \overline{2^{\frac{1}{3}}a^2 + 2^{\frac{2}{3}}ax + 2x^2}}{\overline{a^3 - x^3} \times \overline{g^2x^{-2} + 6g + 12x^2}}$. Therefore, $m - 1$ being $= 1$, m is $= 2$; and, q (the value of $\frac{G}{m}$ when x is $= b$) being a poſitive quantity, y is a maximum when x is $= \frac{a}{2^{\frac{1}{3}}}$.

Taking b equal to a, we have $F = \overline{a - x}^{\,-\frac{1}{3}} = \overline{b - x}^{\,m-1}$, and $G = \dfrac{2x \times \overline{a^3 - x^3}^{\,\frac{1}{3}} - 2x^2}{\overline{a^2 + ax + x^2}^{\,\frac{1}{3}}}$. Therefore, $m - 1$ being $= -\frac{1}{3}$, m is $= \frac{2}{3}$; and q (the value of $\frac{G}{m}$ when x is $= b$) being a negative quantity, y is minimum when x is $= a$.

When m comes out an odd poſitive number, or a poſitive fraction whoſe numerator and denominator are both odd numbers, the ordinate correſponding to the abſciſſa found by our proceſs, inſtead of being a maximum or minimum, paſſes through a point of contrary flexure; and the tangent, at that point, is parallel to the baſe, or coincides with the ordinate, according as m is greater or leſs than 1.

We have all along ſuppoſed, that the curve is continued on both ſides of the ordinate determined as above: if the curve be not ſo continued, either m will be a fraction whoſe numerator is an odd number and denominator an even number, or G will not be real both when x is greater and when leſs than b.

If G be real both when x is greater and when leſs than b; or, d being the value of the ordinate correſponding to the abſciſſa whoſe value is b, if $G^{\frac{1}{m}}$ be real both when y is greater and when leſs than d; then, m being ſuch a fraction as we laſt deſcribed, the curve will be reflected from the ſaid ordinate, either touching it or forming a cuſpis there, according as m is leſs or greater than 1.—If m be greater than 1, both branches of the curve will have their convexity towards each other, and the tangent at the cuſpis will be parallel to the baſe.

If (q being real) neither G be real both when x is greater and when leſs than b, nor $G^{\frac{1}{m}}$ real both when y is greater and when leſs than d; then, m being any number whatever, the curve will form ſuch a cuſpis, at the ordinate (d), that the convexity of one branch of the curve will be towards the concavity of the other, and the tangent at the point of reflection will be parallel to the baſe or coincide with the ordinate, according as m is greater or leſs than 1.

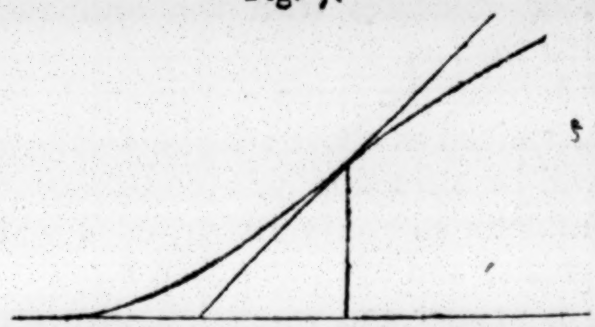
Thus, perhaps, we obtain a more perfect and ſatisfactory knowledge of the article under conſideration, than can be obtained by any other method whatſoever.

We have mentioned how the points of contrary flexure will be determined when the tangent at ſuch point is parallel to the baſe or coincides with the ordinate; let us now enquire how ſuch points may be found in all other poſitions of the tangent.—This buſineſs, after what is ſaid above, will be attended with no great difficulty: for it is eaſy to prove, that the quotient of the ordinate divided by the ſubtangent is a maximum,

(as

Fig. 7.

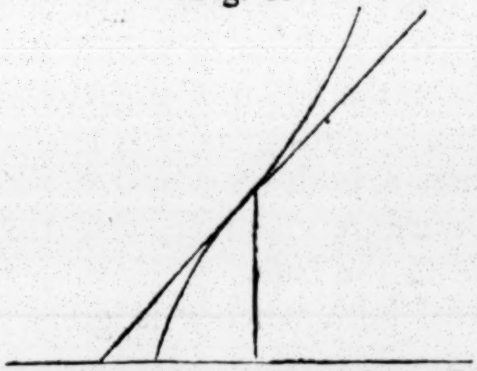
Fig. 7.



(as in Fig. 7.) or a minimum (as in Fig. 8.) when the ordinate paſſes through a point of contrary flexure, and the tangent at ſuch point is oblique to the baſe and ordinate: which quotient is equal to $[x\perp y]$, as appears by what we have ſaid concerning tangents.

Fig. 8.

Fig. 8.



Therefore the points of contrary flexion in curve lines (when the tangent at ſuch point is oblique to the baſe and ordinate) may be readily found by enquiring when $[x\perp y]$ is a maximum or a minimum. And, from what is done above, it evidently follows, that, writing $[x\perp\!\!\perp y]$ for the value of the quotient of $[x\perp y] - [v\perp u]$ divided by $x-v$, when v is therein taken equal to x,

$$[x\perp\!\!\perp y] \text{ will be } = 0, \text{ or } \frac{1}{[x\perp\!\!\perp y]} = 0,$$

when y correſponds to the point ſought.

Moreover, ſuppoſe β to be a value of x in either of thoſe equations, and n to be ſuch that neither $\dfrac{[x\perp\!\!\perp y]}{n\times\overline{\beta-x}^{\,n-1}}$ $(=\text{H})$ nor its reciprocal ſhall vaniſh when x is $=\beta$; alſo ſuppoſe H to be real both when x is greater and when leſs than β. Then if n be an even poſitive number, or a poſitive fraction whoſe numerator is an even number and denominator an odd number, the ordinate correſponding to the abſciſſa β ſhall paſs through a point of contrary flexure; but, if n be an odd poſitive number, or a poſitive fraction whoſe numerator and denominator are both odd numbers, ſuch ordinate ſhall be a maximum or a minimum.

If H be real both when x is greater and when leſs than β; or, δ being the value of $[x \perp y]$ when x is equal to β, if $H^{\frac{1}{n}}$ be real both when $[x \perp y]$ is greater and when leſs than δ; then, n being a poſitive fraction whoſe numerator is an odd number and denominator an even number, the ordinate correſponding to the abſciſſa β will paſs through a cuſpis: which will be ſuch, that both branches of the curve will have their convexity towards each other, and (δ and its reciprocal being ſuppoſed finite) the tangent at the point of reflection will be oblique to the baſe and ordinate.

If (the value of H when x is $= \beta$ being real) neither H be real both when x is greater and when leſs than β, nor $H^{\frac{1}{n}}$ real both when $[x \perp y]$ is greater and when leſs than δ; then, n being any number whatever, the curve will form ſuch a cuſpis at the ordinate correſponding to the abſciſſa β, that the convexity of one branch of the curve will be towards the concavity of the other, and (δ and its reciprocal being finite) the tangent at the point of reflection will be oblique to the baſe and ordinate.

We now proceed to ſay ſomething of the curvature of lines.

Fig. 9.

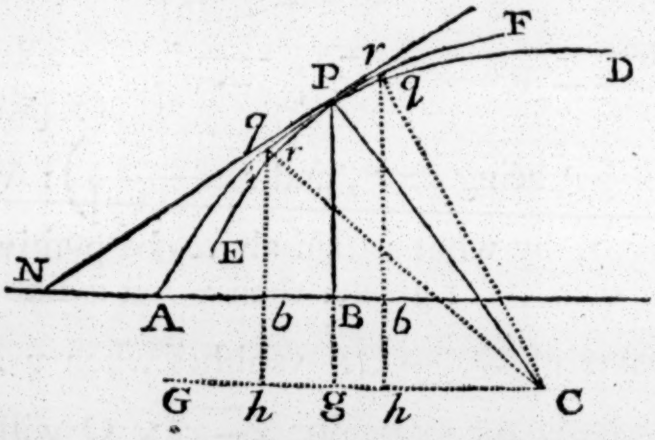


Fig. 9.

Let APD be any curve line; and ſuppoſe the circle EPF, deſcribed about the center C, to be the circle of curvature at the point P.

Now, the curvature being uniform in the circle only, the curvature of APD will (except when it is a maximum or a minimum) increaſe on one ſide of P, and decreaſe on the other ſide of that point.—Suppoſe the curvature of APD to decreaſe from P towards A, and to increaſe from P towards D. Then will the arc PE of the circle of curvature fall within the curve PA, and the arc PF (of the ſame circle) will fall without the curve PD. Therefore, drawing the radius C*r*, interſecting the curve APD in *q*, C*q* will be greater or leſs than C*r*, according as *r* is on this or on that ſide of P: and conſequently the value of C*q*—C*r*, or of $\overline{Cq}^2 - \overline{Cr}^2$, will be poſitive or negative, according as the abſciſſa correſponding to the point *q* is leſs or greater than the abſciſſa correſponding to the point P, ſuch abſciſſas being meaſured from A upon the baſe AB.

Draw the ordinates PB, *qb*; and the tangent PN. Draw, alſo, CG parallel to AB, and continue the ordinates PB, *qb*, to *g* and *h* reſpectively. Call AB, *x*; BP, *y*; A*b*, *v*; *bq*, *u*; BN, *s*; PN, *t*; and the radius CP, (or C*r*,) R.

Then, by ſimilar triangles, we have $Pg = \frac{Rs}{t}$, $Cg = \frac{Ry}{t}$; and conſequently $Bg\ (=bh) = \frac{Rs}{t} - y$, $hq = u - y + \frac{Rs}{t}$, $Ch = x - v + \frac{Ry}{t}$, and $\overline{Cq}^2 - \overline{Cr}^2 \left(= \overline{Ch}^2 + \overline{hq}^2 - \overline{Cr}^2\right) =$

$$\overline{y-u}^2 - \frac{2Rs}{t} \times \overline{y-u} + \frac{R^2s^2}{t^2} + \overline{x-v}^2 + \frac{2Ry}{t} \times \overline{x-v} + \frac{R^2y^2}{t^2} - R^2 = \overline{y-u}^2 + \overline{x-v}^2 + \frac{2Ry}{t \times [x\perp y]} \times \overline{\overline{x-v} \times [x\perp y] - \overline{y-u}}$$

$\left(s^2 + y^2 \text{ being} = t^2, \text{ and } s = \frac{y}{[x\perp y]}\right)$: Which value of $\overline{Cq}^2 - \overline{Cr}^2$ muſt, by what is ſaid above, be poſitive or negative, according as *v* is leſs or greater than *x*: it is likewiſe obſervable, that the value of $\overline{Cq}^2 - \overline{Cr}^2$ vaniſhes when *v* is equal to *x*.

Now the expreſſion $\overline{x-v}^m \times Q$ will vaniſh when *v* is equal to *x*, and be poſitive or negative according as *v* is leſs or greater than *x*, if *m* be an odd poſitive number, or a poſitive fraction whoſe numerator and denominator are both odd numbers; and Q an algebraic expreſſion ſo compoſed of *v* and other quantities, that (q) its value when v is equal to x is finite and poſitive *.

It is obvious therefore, that, (ſuppoſing m and Q to be as here ſpecified,) if

$$\overline{y-u}^{2} + \overline{x-v}^{2} + \frac{2Ry}{t \times [x \perp y]} \times \overline{\overline{x-v} \times [x \perp y] - \overline{y-u}}$$

be aſſumed $= \overline{x-v}^{m} \times Q$, the value of R from thence determined will be the value of, the radius of curvature correſponding to the point P.

When the curvature of APD increaſes from P towards A, and decreaſes from P towards D, the expreſſion $\overline{y-u}^{2} + \overline{x-v}^{2} + \frac{2Ry}{t \times [x \perp y]} \times \overline{\overline{x-v} \times [x \perp y] - \overline{y-u}}$ will be negative or poſitive, according as v is leſs or greater than x; and our aſſumed equation muſt be the ſame as above, except that q muſt be negative inſtead of poſitive.

To find R, let each ſide of the aſſumed equation be divided by $x-v$; by which means we get

$$\overline{y-u} \times [x \mid y] + x-v + \frac{2Ry}{t \times [x \perp y]} \times \overline{[x \perp y] - [x \mid y]} =$$

$\overline{x-v}^{m-1} \times Q$; from whence nothing can be determined by taking v equal to x, both ſides of the equation then vaniſhing. We therefore again divide by $x-v$, and, by ſo doing, find

$$[x \mid y]^{2} + 1 + \frac{2Ry}{t \times [x \perp y]} \times \frac{[x \perp y] - [x \mid y]}{x-v} = \overline{x-v}^{m-2} \times Q.$$

Hence, by taking v equal to x, we have

$$[x \perp y]^{2} + 1 + \frac{Ry}{t \times [x \perp y]} \times [x \perp\!\!\perp y] = 0;$$

the quotient of $[x \perp y] - [x \mid y]$ divided by $x-v$ being then equal to $\frac{[x \perp\!\!\perp y]}{2}$, (Vid. p. 13. and 24.). From which laſt equation we find

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* See note 5. at the end,

$$R = -\frac{t \times [x \perp y] \times \overline{1 + [x \perp y]^2}}{y \times [x \perp\!\!\perp y]} = -\frac{\overline{1 + [x \perp y]^2}^{\frac{3}{2}}}{[x \perp\!\!\perp y]};$$

t being $= \sqrt{1 + [x \perp y]^2} \times \frac{y}{[x \perp y]}$, as appears by what is ſaid above concerning tangents.

If, now, the firſt written value of R be ſubſtituted above, we ſhall have $\frac{1 + [x \perp y]^2}{[x \perp\!\!\perp y]} \times [x \perp\!\!\perp y] - 2 \times \frac{[x \perp y] - [x \mid y]}{x - v} = \overline{x - v}^{m-2} \times Q$; in which equation, it is eaſy to prove, (but I cannot ſtay to do it here,) that, q and its reciprocal being finite, m will in general be equal to 3: which being agreeable to our ſuppoſition, it follows, that R is rightly determined; and q muſt of neceſſity be poſitive or negative, according as the curvature of PA decreaſes or increaſes from P towards A.

EXAMPLE. *Suppoſe the curve* APD *to be the parabola, whoſe equation is* $ax = y^2$.

Then $ax - av$ will be $= y^2 - u^2$; from whence, by dividing by $x - v$, we get $a = \overline{y + u} \times [x \mid y]$; and hence, by taking v equal to x, we have $a = 2y \times [x \perp y]$. Therefore $[x \perp y]$ is $= \frac{a}{2y}$, and $\overline{1 + [x \perp y]^2}^{\frac{3}{2}} = \overline{\frac{4y^2 + a^2}{8y^3}}^{\frac{3}{2}}$. Moreover $[x \perp y] - [v \perp u]$ will be $= \frac{a}{2y} - \frac{a}{2u}$, and the quotient of $[x \perp y] - [v \perp u]$ divided by $x - v$ will be equal to $\frac{-a}{2uy} \times [x \mid y]$: Therefore $[x \perp\!\!\perp y]$, the value of that quotient when v is equal to x, will be equal to $\frac{-a[x \perp y]}{2y^2} = \frac{-a^2}{4y^3}$. Conſequently, by writing $\overline{\frac{4y^2 + a^2}{8y^3}}^{\frac{3}{2}}$ and $\frac{-a^2}{4y^3}$ inſtead of $\overline{1 + [x \perp y]^2}^{\frac{3}{2}}$ and $[x \perp\!\!\perp y]$ reſpectively in the equation $R = -\frac{\overline{1+[x\perp y]^2}^{\frac{3}{2}}}{[x\perp\!\!\perp y]}$, we find $R = \frac{\overline{4y^2+a^2}^{\frac{3}{2}}}{2a^2}$.

Our deſign now leads us to conſider the *menſuration* of *curvilinear ſpaces.*

It is manifeſt that in Fig. 10. and 11. the curvilinear area BPqb is leſs or greater than the rectangle BP × Bb, according as Ab is leſs or greater than AB.

Therefore, calling AB, x; BP, y; Ab, v; bq, (parallel to BP,) u; and the areas HBPh, Hbqh, z and w reſpectively: $\overline{z-w}$ will be always leſs than $y\times\overline{x-v}$, when v is taken either greater or leſs than x; and conſequently the expreſſion $y\times\overline{x-v}-\overline{z-w}$ will be always poſitive when v is ſo taken.

Fig. 10. Fig. 10.

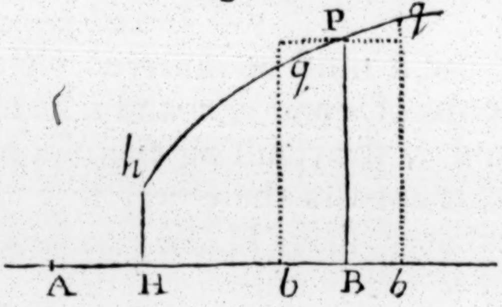
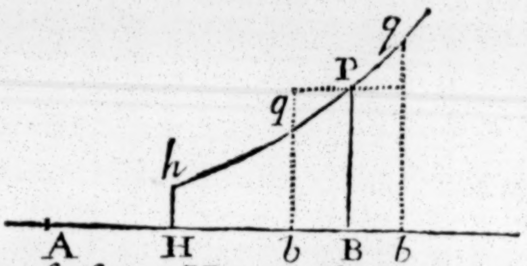


Fig. 11. Fig. 11.



Moreover, if the ordinates decreaſe from H towards B, $\overline{z-w}$ will be always greater than $y\times\overline{x-v}$, when v is taken either greater or leſs than x; and $y\times\overline{x-v}-\overline{z-w}$ will be alwys negative when v is ſo taken.

Now, the expreſſion $y\times\overline{x-v}-\overline{z-w}$ being ſimilar to that which we found equal to qr in p. 10. we may, by what is ſaid in p. 11. and 12. aſſume

$$y\times\overline{x-v}-\overline{z-w}=\overline{x-v}^2\times Q,$$

Q being an expreſſion ſo compoſed of v and other quantities, that neither it nor its reciprocal vaniſhes when v is equal to x; and the value of z determined by means of that equation ſhall be equal to the area HBP*b*. As to Q, it may be farther obſerved, that its value, when v is equal to x, muſt of neceſſity be poſitive or negative, according as the ordinates increaſe or decreaſe from H towards B.

To find z, let each ſide of the aſſumed equation be divided by $x - v$; by which means we get

$$y - [x \mid z] = \overline{x - v} \times Q,$$

$[x \mid z]$ being put for the quotient of $z - w$ divided by $x - v$.

Conſequently, by taking v equal to x, and writing $[x \perp z]$ for the value of $[x \mid z]$ when v is ſo taken, we have

$$y - [x \perp z] = 0, \text{ and } [x \perp z] = y.$$

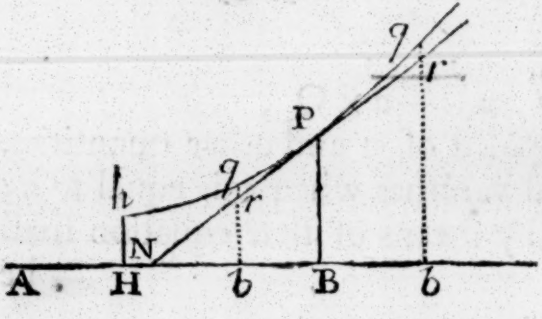
Now it muſt be obſerved, that, if in the value of z expreſſed in terms of x and y, v and u be reſpectively ſubſtituted inſtead of x and y, the reſulting expreſſion will be equal to w.

It appears therefore, that, in computing the area of the curvilinear ſpace HBP*b*, our buſineſs is to find ſuch a function of x, that, a ſimilar function of v being ſubtracted therefrom, and the remainder divided by $x - v$, the quotient, when v is therein taken equal to x, ſhall be equal to the ordinate y.

Having found ſuch function of x, let ſome quantity (k), (that is independent of x,) be added thereto, and the ſum ſhall be equal to z; and k will be determined by ſubſtituting, in that equation, certain values of x and z that are known to correſpond to each other.——Such addition of the quantity k is neceſſary, becauſe, H*h* being at any diſtance whatever from A, the quotient of $z - w$ divided by $x - v$ will be the ſame; and therefore ſome part of the expreſſion denoting the value of z may depend on that diſtance, and not on x; which part of that expreſſion (when ſuch there is) will be known by proceeding as above directed.

Fig. 12.

Fig. 12.



If, inſtead of computing the area of a curve, we would find the length thereof; we are to conſider, that, when the curve is convex towards the baſe, and the angle BPN, made by the ordinate BP and tangent PN, is acute, (as in Fig. 12.) the ordinate *bq* being drawn, interſecting NP in *r*, P*r* (by an *Axiom* of ARCHIMEDES) will be greater or leſs than P*q*, according as A*b* is leſs or greater than AB. Therefore, calling AB, BP, A*b*, *bq*, as before; and the parts *b*P, *bq* of the curve, z' and w' reſpectively: $z'-w'$ will be always leſs than $\frac{t}{s}\times\overline{x-v}$, when v is taken either greater or leſs than x; and conſequently the expreſſion $\frac{t}{s}\times\overline{x-v}-\overline{z'-w'}$ will be always poſitive when v is ſo taken; s and t denoting the ſubtangent BN and tangent PN reſpectively.

Fig. 13.

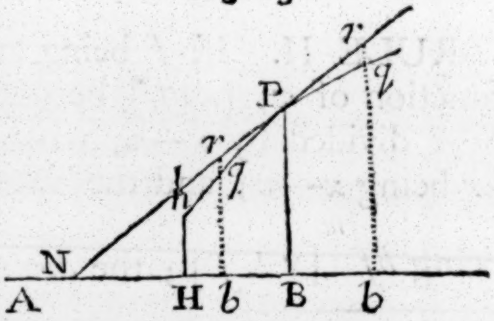


Fig. 13.

Moreover, when the curve is concave towards the baſe, and the angle BPN is acute, (as in Fig. 13.) $z'-w'$ will be always greater than $\frac{t}{s}\times\overline{x-v}$, when v is taken either greater or leſs than x; and $\frac{t}{s}\times\overline{x-v}-\overline{z'-w'}$ will be always negative when v is ſo taken.

Now, the expreſſion $\frac{t}{s}\times\overline{x-v}-\overline{z'-w'}$ being ſimilar to $y\times\overline{x-v}-\overline{z-w}$, it will appear, by proceeding with the former as we juſt now did with the latter, that, in the ſame manner as z may be found from y, z' may be found from $\frac{t}{s}$ or its equal $\sqrt{1+[x\perp y]^2}$, which, by what is ſaid above, will be known when the equation of the curve is given.

It appears then, that, in computing the area or length of a curve, when its equation is given, we ſhall have the *diviſor* $(x-v)$, and the value of the *quotient* when v is equal to x, given; to find the correſpondent *dividend*, which muſt be ſuch a *reſidual*, that one part thereof, called the *prime member*, ſhall be ſome function of x, and the other part a ſimilar function of v: which prime member, (corrected by the addition of the quantity k, as above-mentioned,) ſhall be the value of ſuch area or length, according as the ſaid value of the quotient is equal to y or $\sqrt{1+[x\perp y]^{2}}$.—Rules for aſſigning ſuch dividend may be deduced from our theorems in p. 5. and the notes at the end of this diſcourſe.

RULE I. The *diviſor* being $x-v$; and the value of the *quotient*, when v is $=x$, being bx^{m}; the *dividend* muſt be equal to $\frac{b}{m+1}\times\overline{x^{m+1}-v^{m+1}}$.

RULE II. If, f being any function of x, and e a ſimilar function of v, $[x\perp f]$ be put for the value of the quotient of $f-e$ divided by $x-v$, when v is equal to x: then, the *diviſor* being $x-v$; and the value of the *quotient*, when v is $=x$, being $bf^{m}\,[x\perp f]$; the *dividend* muſt be equal to $\frac{b}{m+1}\times\overline{f^{m+1}-e^{m+1}}$.

RULE III. If, F and f being any functions of x, and E and e ſimilar functions of v reſpectively, $[x\perp F]$ and $[x\perp f]$ be reſpectively put for the values of the quotients $\overline{F-E}\div\overline{x-v}$ and $\overline{f-e}\div\overline{x-v}$, when v is equal to x: then, the *diviſor* being $x-v$; and the value of the *quotient*, when v is $=x$, being $b\times\overline{F\,[x\perp f]+f\,[x\perp F]}$; the *dividend* muſt be equal to $b\times\overline{Ff-Ee}$.

Other rules, for the like purpoſe, may be derived from the ſame ſource; but let theſe ſuffice for the preſent.

As $\overline{f+k}-\overline{e+k}$ is $=f-e$, it is plain, that, the dividend remaining the ſame, the prime member thereof may not only be equal to f, but it may alſo be equal to f increaſed or diminiſhed by any quantity that is independent of x. Hence, therefore, ſore, appears the neceſſity of correcting (in the manner above ſhewn) the prime member of ſuch dividend, from ſome condition of the propoſition in whoſe inveſtigation an expreſſion of that kind may ariſe.

EXAMPLE. *Suppoſe the curve to be the ſemicubical parabola,* Fig. 11. *whoſe equation is* $ax^2 = y^3$. Then will y be $= a^{\frac{1}{3}}x^{\frac{2}{3}}$.

Now, by Rule I. $x - v$ being the diviſor, and $a^{\frac{1}{3}}x^{\frac{2}{3}}$ the value of the quotient when v is equal to x; the dividend muſt be $= \frac{3}{5}a^{\frac{1}{3}} \times \overline{x^{\frac{5}{3}} - v^{\frac{5}{3}}}$. Therefore, $\frac{3}{5}a^{\frac{1}{3}}x^{\frac{5}{3}}$ being the prime member of the ſaid dividend, z will be $= \frac{3}{5}a^{\frac{1}{3}}x^{\frac{5}{3}} + k$, the area HBP*h*.

To find k, let the diſtance AH be called d: then, z being $= 0$ when x is $= d$, we have $\frac{3}{5}a^{\frac{1}{3}}d^{\frac{5}{3}} + k = 0$; therefore k is $= -\frac{3}{5}a^{\frac{1}{3}}d^{\frac{5}{3}}$, and the area HBP*h* is $= \frac{3}{5}a^{\frac{1}{3}}x^{\frac{5}{3}} - \frac{3}{5}a^{\frac{1}{3}}d^{\frac{5}{3}}$.

Moreover, (y being $= a^{\frac{1}{3}}x^{\frac{2}{3}}$), $[x \perp y]$ will be $= \frac{2}{3}a^{\frac{1}{3}}x^{-\frac{1}{3}}$ and $\sqrt{1 + [x \perp y]^2} = \frac{\sqrt{9x^{\frac{2}{3}} + 4a^{\frac{2}{3}}}}{3x^{\frac{1}{3}}} = \frac{f^{\frac{1}{2}}[x \perp f]}{18}$; f being $= 9x^{\frac{2}{3}} + 4a^{\frac{2}{3}}$, $e = 9v^{\frac{2}{3}} + 4a^{\frac{2}{3}}$, and $[x \perp f]$ $(= 6x^{-\frac{1}{3}})$ the value of the quotient of $f - e$ divided by $x - v$, when v is $= x$.

Now, by Rule II. the diviſor being $x - v$, and $\frac{f^{\frac{1}{2}}[x \perp f]}{18}$ being the value of the quotient when v is $= x$; the dividend muſt be $= \frac{1}{27} \times \overline{f^{\frac{3}{2}} - e^{\frac{3}{2}}}$. Therefore, $\frac{f^{\frac{3}{2}}}{27}$, or its equal $\frac{\overline{9x^{\frac{2}{3}} + 4a^{\frac{2}{3}}}^{\frac{3}{2}}}{27}$, being the prime member of the ſaid dividend, z'

will be $= \frac{1}{27} \times \overline{9x^{\frac{2}{3}}+4a^{\frac{2}{3}}}^{\frac{3}{2}} + k$, the length of the curve bP.

—But, z' being $= 0$ when x is $= d$, $\frac{1}{27} \times \overline{9d^{\frac{2}{3}}+4a^{\frac{2}{3}}}^{\frac{3}{2}} + k$ will be $= 0$; therefore k will be $= -\frac{1}{27} \times \overline{9d^{\frac{2}{3}}+4a^{\frac{2}{3}}}^{\frac{3}{2}}$, and the arc $bP = \frac{1}{27} \times \overline{9x^{\frac{2}{3}}+4a^{\frac{2}{3}}}^{\frac{3}{2}} - \frac{1}{27} \times \overline{9d^{\frac{2}{3}}+4a^{\frac{2}{3}}}^{\frac{3}{2}}$.

I ſhall ſay no more, at preſent, of the quadrature and rectification of curves; but proceed to take notice of ſome propoſitions of another kind.

Suppoſe V and v to denote the velocities generated in a moving body, in the times T and t reſpectively, by a variable force, acting continually upon that body, in any manner whatever: then, if, at the end of the time T, the force is ſuch, that it would, if uniformly continued, generate in a given time a velocity denoted by F; the reſidual $V-v$ will, it is obvious, be leſs or greater than $F \times \overline{T-t}$, according as the force increaſes or decreaſes, its increaſe or decreaſe being ſuppoſed continual. Conſequently the expreſſion $F \times \overline{T-t} - \overline{V-v}$ will be always poſitive when F increaſes, and always negative when F decreaſes; t being any time leſs or greater than T (between certain limits).

Now, the expreſſion $F \times \overline{T-t} - \overline{V-v}$ being ſimilar to that which we found equal to qr in p. 10. we may, by what follows preſently thereafter, aſſume

$$F \times \overline{T-t} - \overline{V-v} = \overline{T-t}^{2} \times Q,$$

Q being an expreſſion ſo compoſed of t and other quantities, that neither it nor its reciprocal vaniſhes when t is taken equal to T. And it is obſervable, that the value of Q, when t is ſo taken, muſt of neceſſity be poſitive or negative, according as F increaſes or decreaſes.

From the aſſumed equation we get, by diviſion,

F—

$$F - [T \mid V] = \overline{T - t} \times Q,$$

$[T \mid V]$ being put for the quotient of $V - v$ divided by $T - t$.

Conſequently, by taking t equal to T, and writing $[T \perp V]$ for the value of $[T \mid V]$ when t is ſo taken, we have

$$F - [T \perp V] = 0, \text{ and } F = [T \perp V].$$

If S and s denote the ſpaces paſſed over, in the times T and t reſpeictvely, by a body moving with a variable velocity, in any manner whatever; ſuch velocity correſponding to the time T being denoted by V, the reſidual $S - s$ will be leſs or greater than $V \times \overline{T - t}$, according as (the velocity denoted by) V is accelerated or retarded: and, conſequently, the expreſſion $V \times \overline{T - t} - \overline{S - s}$ will be always poſitive when V is accelerated, and always negative when V is retarded; t being any time greater or leſs than T (between certain limits).

Now, the expreſſion $V \times \overline{T - t} - \overline{S - s}$ being ſimilar to $F \times \overline{T - t} - \overline{V - v}$, we may proceed with the former as we juſt now did with the latter; and, by ſo doing, we ſhall find $V = [T \perp S]$, the value of the quotient of $S - s$ divided by $T - t$ when t is therein taken equal to T.

The fundamental theorems in the doctrine of motion, are theſe we have juſt now inveſtigated, viz. $F = [T \perp V]$ and $V = [T \perp S]$; and I purpoſe, before I put an end to this diſcourſe, to give an inſtance of their uſe, in finding the law of centripetal force neceſſary to cauſe a body to deſcribe a given curve, about the center towards which that force continually tends.

Let it firſt be obſerved, that, by the meaſure of a force with reſpect to any particular direction, muſt be underſtood, the velocity which that force, uniformly continued, is capable of generating in a given body, in a given time, in that particular direction. Therefore V and v being the velocities of a given body, in a given direction, at the end of the times T and t reſpectively; $[T \perp V]$, the value of the quotient of $V - v$ divided by $T - t$ when t is therein taken equal to T, ſhall, by what is ſaid above, be the meaſure of that force, with reſpect to that given direction.

Fig. 14.

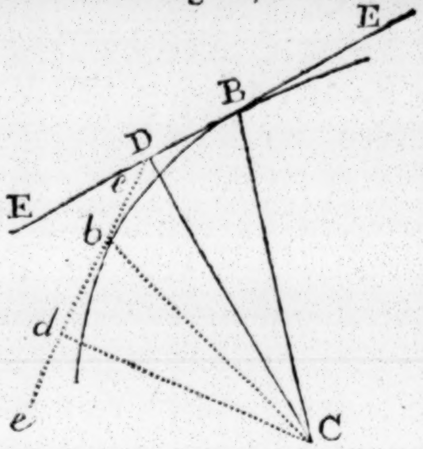


Fig. 14.

This being premiſed, let C be the center, towards which the body (M) is continually urged; and ſuppoſe B and *b* to be any two points in the curve; BD, *bd*, tangents to the curve at thoſe points; and CD, C*d*, perpendiculars upon the ſaid tangents. When the body is at B, let the meaſure of the centripetal force acting thereon in the direction BC be denoted by F: and let T and *t* be reſpectively put for the times the body has been in motion when it arrives at B and *b*; and V and *v* for its velocities at thoſe points, in the directions BE, *be* reſpectively. Call CB, S; C*b*, *s*; CD, P; C*d*, *p*; BD, *q*; and the circulatory velocity of the body, (i. e. its velocity in a direction at right angles to the ray drawn thereto from the center of force,) *u*.

$$\text{Then } S : P :: V : u\text{; therefore } u \text{ is } = \frac{PV}{S}.$$

Now, if the centripetal force ceaſed to act on the body when it arrived at B, it would proceed along the tangent BD; and P and V would remain invariable. Therefore $-\frac{PV[T\perp S]}{S^2}$, the value of the quotient of $\frac{PV}{S} - \frac{PV}{s}$ divided by T—*t* when *t* is therein taken equal to T, is (by what is ſaid above) the meaſure of a force (G) which, with reſpect to any certain direction, would be capable of accelerating or retarding the velocity of any body equal to M, at the ſame rate as the circulatory velocity of M would be accelerated or retarded if the centripetal force ceaſed to act as above-mentioned.

But, the centripetal force continuing to act upon the body, it will proceed along the curve; and P, S, and V will all be variable. Therefore $\frac{V[T\perp P]}{S} + \frac{P[T\perp V]}{S} - \frac{PV[T\perp S]}{S^2}$, the value of the quotient of $\frac{PV}{S} - \frac{pv}{s}$ divided by $T-t$ when t is therein taken equal to T, is the meaſure of a force (H) which, with reſpect to any certain direction, would be capable of accelerating or retarding the velocity of any body equal to M, at the ſame rate as u is actually accelerated or retarded while the body deſcribes the curve B*b*; $[T\perp P]$ being put (agreeable to our notation) for the value of the quotient $\overline{P-p}\div\overline{T-t}$ when t is taken as juſt now mentioned.

Conſequently, taking the meaſure of the force G from that of the force H, the remainder will be the true meaſure of the efficacy of the centripetal force to accelerate or retard u.—— But the centripetal force has no power to accelerate or retard the velocity of the body in a direction at right angles to that in which the ſaid force acts: Therefore that remainder, viz.

$$\frac{V\,[T\perp P]}{S} + \frac{P\,[T\perp V]}{S} \text{ muſt be } = 0.$$

$$\text{Hence } V\,[T\perp P] + P\,[T\perp V] = 0:$$

Which equation is derived from

$$V \times \frac{P-p}{T-t} + p \times \frac{V-v}{T-t} = 0, \text{*}$$

by taking t equal to T; p being then $= P$, and $v = V$.

Therefore $V \times \overline{P-p} + p \times \overline{V-v} = PV - pv$ is $= 0$, and $PV = pv$.

Whence it appears, that the velocity of the body will always be as the reciprocal of the perpendicular from the center of force upon the tangent.

Let b and c be the reſpective values of p and v at any term of time: Then PV being $= bc$, V is $= \frac{bc}{P}$, $v = \frac{bc}{p}$, $\overline{V-v}\div\overline{T-t}$ $= bc \times \overline{\frac{1}{P} - \frac{1}{p}} \div \overline{T-t}$, and $[T\perp V] = -\frac{bc\,[T\perp P]}{P^2}$.

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* See note 4. at the end.

Moreover, $S : q :: V : \frac{qV}{S}$, the velocity wherewith the ray CB decreaſes or increaſes; which, by what is done above, is $= \mp [T \perp S]$: Therefore $\frac{q}{S}$ is $= \mp \frac{[T \perp S]}{V} = \mp \frac{P\,[T \perp S]}{bc}$.

Alſo, by the reſolution of forces,

$S : q :: F : \frac{qF}{S}$, the meaſure of the efficacy of the centripetal force in the direction BE; which, by what is above explained, muſt be $= \pm [T \perp V] = \mp \frac{bc\,[T \perp P]}{P^2}$:

Therefore F is $= \mp \frac{S}{q} \times \frac{bc\,[T \perp P]}{P^2} = \frac{b^2c^2\,[T \perp P]}{P^3\,[T \perp S]}$.

EXAMPLE I. *Suppoſe the curve to be a circle deſcribed about the center* C *with the radius* R: Then, P and S being each equal to R, $\frac{[T \perp P]}{[T \perp S]}$ will be $= 1$; and (*b* being alſo $=$ R) F, it is manifeſt, will be $= \frac{c^2}{R}$.

Hence all Mr. HUYGENS's theorems relating to a centrifugal force and circular motion may be eaſily deduced.

Seeing F is here $= \frac{c^2}{R}$, it is evident, that the velocity in a circle whoſe radius is S will be $F^{\frac{1}{2}}S^{\frac{1}{2}}$. Moreover, by what is done above, $\left(\frac{bc}{P}\right)$ the velocity in any curve at the diſtance S from the center of force is $\frac{F^{\frac{1}{2}}P^{\frac{1}{2}}\,[T \perp S]^{\frac{1}{2}}}{[T \perp P]^{\frac{1}{2}}}$.

It follows therefore, that, the body moving in any curve whatever, its velocity at the diſtance S from the center of force, is to the velocity wherewith the ſame body might revolve in a circle, about the ſame center, at the ſame diſtance therefrom, as $\frac{F^{\frac{1}{2}}P^{\frac{1}{2}}\,[T\perp S]^{\frac{1}{2}}}{[T\perp P]^{\frac{1}{2}}}$ to $F^{\frac{1}{2}}S^{\frac{1}{2}}$, i. e. as $\frac{P^{\frac{1}{2}}}{S^{\frac{1}{2}}}$ to $\frac{[T\perp P]^{\frac{1}{2}}}{[T\perp S]^{\frac{1}{2}}}$.

EXAMPLE II. *Suppoſe the curve to be the conical parabola, and* C *the focus thereof:* Then, by the nature of the curve, *a* being the parameter, P is $= \frac{1}{2}a^{\frac{1}{2}}S^{\frac{1}{2}}$.

Therefore $[T\perp P]$ is $= \frac{1}{4}a^{\frac{1}{2}}S^{-\frac{1}{2}}\,[T\perp S]$; and conſequently, by ſubſtitution, F is found $= \frac{2b^2c^2}{aS^2}$.

Moreover, $\frac{P^{\frac{1}{2}}}{S^{\frac{1}{2}}}$ is to $\frac{[T\perp P]^{\frac{1}{2}}}{[T\perp S]^{\frac{1}{2}}}$ as $\sqrt{2}$ to 1, which is the conſtant ratio of the velocity of the body in the parabola, to the velocity wherewith it might deſcribe a circle at the ſame diſtance from the center of force.

It may be obſerved, that our method of aſſuming an equation in order to give one member thereof a certain property with reſpect to being poſitive or negative in certain caſes, ſomewhat reſembles the method made uſe of in reſolving certain Diophantine problems:—but ſuch reſemblance, I preſume, will not be deemed a fault. Moreover, with reſpect to the proceſs conſequent to ſuch aſſumption, it may be remarked, that we aſſign the quotient ariſing by dividing one reſidual by another, and afterwards take the value of that quotient in one particular caſe, which is commonly a more troubleſome operation than that which the method of fluxions requires in reſolving the ſame problem.—But let it likewiſe be remarked, that the trouble of aſſigning the general value of ſuch quotient may be avoided, (without bringing the leaſt obſcurity upon the proceſs,) by aſſigning only the required particular value thereof; which may be done, (as will be ſhewn in our treatiſe,) with great facility, by brief rules deduced, for that purpoſe, from a conſideration of the manner of obtaining ſuch general value: and, conſequently our method (which, perhaps, is more comprehenſive than any other) will be thereby rendered as conciſe as the fluxionary method.

The reader has now before him enough, I hope, to ſatisfy his curioſity, for the preſent, with reſpect to the *Reſidual Analyſis*:—concerning which he may, ere long, be farther ſatisfied, if he pleaſes to conſult my intended treatiſe, when publiſhed; wherein will be conſidered a great variety of articles, which are here deſignedly omitted; and ſuch as are here but ſlightly touched upon, will be there more fully explained. In particular, that treatiſe will contain a general theorem for the reſolution of the celebrated *Iſoperimetrical Problems*: and ſeveral diſquiſitions relating to *Equilibriums*; the *Powers of Machines*, or *Engines*, moved by the wind, water, or otherwiſe; the *Solids of leaſt Reſiſtance*; the *Curves of ſwifteſt Deſcent*; and the *Motion of Bodies affected by projectile and centripetal Forces*, &c.

N O T E S.

1. $\frac{x^{\frac{m}{n}} - v^{\frac{m}{n}}}{x-v}$ being $= x^{\frac{m}{n}-1} \times \frac{1+\frac{v}{x}+\left.\overline{\frac{v}{x}}\right|^{2} + \left.\overline{\frac{v}{y}}\right|^{3} \quad (m)}{1+\left.\overline{\frac{v}{x}}\right|^{\frac{m}{n}} + \left.\overline{\frac{v}{x}}\right|^{\frac{2m}{n}} + \left.\overline{\frac{v}{x}}\right|^{\frac{3m}{n}} \quad (n)}$, (as is obſerved in page 5.) we have, by writing $1+x$ and $1+y$ inſtead of x and v reſpectively,

$$\frac{\overline{1+x}^{\frac{m}{n}} - \overline{1+y}^{\frac{m}{n}}}{\overline{1+x} - \overline{1+y}} \left(= \frac{\overline{1+x}^{\frac{m}{n}} - \overline{1+y}^{\frac{m}{n}}}{x-y} \right) =$$

$$\overline{1+x}^{\frac{m}{n}-1} \times \frac{1 + \frac{1+y}{1+x} + \left.\overline{\frac{1+y}{1+x}}\right|^{2} \quad (m)}{1 + \left.\overline{\frac{1+y}{1+x}}\right|^{\frac{m}{n}} + \left.\overline{\frac{1+y}{1+x}}\right|^{\frac{2m}{n}} \quad (n)}.$$

2. As we can aſſign the quotient of $x^{\frac{m}{n}} - y^{\frac{m}{n}}$ divided by $x-y$, and as $ax^{\frac{p}{q}} + bx^{\frac{r}{s}}$ &c. $- ay^{\frac{p}{q}} + by^{\frac{r}{s}}$ &c. is $= a \times \overline{x^{\frac{p}{q}} - y^{\frac{p}{q}}} + b \times \overline{x^{\frac{r}{s}} - y^{\frac{r}{s}}}$ &c. it is obvious, that we can readily aſſign the quotient of $ax^{\frac{p}{q}} + bx^{\frac{r}{s}}$ &c. $- ay^{\frac{p}{q}} + by^{\frac{r}{s}}$ &c. divided by $x-y$.

3. If $\frac{y^{\frac{m}{n}} - u^{\frac{m}{n}}}{y-u}$ be put $=$ G, and $\frac{y-u}{x-v} = g$; then, $y-u$ being $= gx - gv$, $\frac{y^{\frac{m}{n}} - u^{\frac{m}{n}}}{gx-gv}$ will be $=$ G, and $\frac{y^{\frac{m}{n}} - u^{\frac{m}{n}}}{x-v} = \mathrm{G}g$.

4. $yz-uw$ is evidently $= z \times \overline{y-u} + u \times \overline{z-w}$; therefore

$$\frac{yz-uw}{x-v} \text{ is } = z \times \frac{y-u}{x-v} + u \times \frac{z-w}{x-v}.$$

It may be obſerved, that this eaſy article anſwers the ſame purpoſe, in our *Analyſis*, as does the aſſigning the *fluxion of a Rectangle*, in the *Doctrine of Fluxions.*

5. Q being an algebraic expreſſion ſo compoſed of v and other quantities, that neither it nor its reciprocal vaniſhes when v is therein taken equal to x; the ſaid expreſſion, when v is greater or leſs than x, (between certain limits,) will, if it does not become imaginary, be poſitive or negative, according as (q) the value of Q, when v is equal to x, is poſitive or negative.

For, let A and B be poſitive quantities; and ſuppoſe that either Q, or $\frac{1}{\mathrm{Q}}$, is $= 0$, when v is $= x + \mathrm{A}$; and likewiſe that Q, or $\frac{1}{\mathrm{Q}}$, is $= 0$, when v is $= x - \mathrm{B}$; but that neither Q, nor $\frac{1}{\mathrm{Q}}$, is $= 0$, when v is $= x + a$, or when v is $= x - b$, a and b being any poſitive quantities leſs than A and B reſpectively: then, taking v of any value, either greater or leſs than x, between the limits $x + \mathrm{A}$ and $x - \mathrm{B}$, Q, it is obvious, will be always poſitive or always negative; and, conſequently, poſitive or negative, according as q is poſitive or negative.

Moreover, m being an even poſitive number, or a poſitive fraction whoſe numerator is an even number and denominator an odd number, $\overline{x-v}^{\,m}$ vaniſhes when v is $= x$, and is always poſitive when v is either greater or leſs than x. Therefore, if m be as juſt now ſpecified, and Q denote ſome algebraic expreſſion ſo compoſed of v and other quantities, that neither q nor its reciprocal is $= 0$; $\overline{x-v}^{\,m} \times \mathrm{Q}$ will be $= 0$, when v is $= x$; and, v being either greater or leſs than x, (between certain limits,) the ſame expreſſion ($\overline{x-v}^{\,m} \times \mathrm{Q}$) will, ſuppoſing Q not to become imaginary, be poſitive or negative, according as q is poſitive or negative.

Furthermore, m being an odd poſitive number, or a poſitive fraction whoſe numerator and denominator are both odd numbers, $\overline{x-v}^{\,m}$ vaniſhes when v is $= x$, and is poſitive or negative, according as v is leſs or greater than x. Therefore, if m be as here ſpecified, and Q denote ſome algebraic expreſſion ſo compoſed of v and other quantities, that neither q, nor $\frac{1}{q}$, is $= 0$; $\overline{x-v}^{\,m} \times Q$ will be $= 0$, when v is $= x$: and, q being poſitive, the value of $\overline{x-v}^{\,m} \times Q$, ſuppoſing Q not to become imaginary, will be poſitive or negative, according as v (when taken between certain limits) is leſs or greater than x; or, q being negative, the value of $\overline{x-v}^{\,m} \times Q$ (ſtill ſuppoſing Q not to become imaginary) will be poſitive or negative, according as v (when taken between certain limits) is greater or leſs than x.

6. The numerator $\overline{a^3-x^3}|^{\frac{2}{3}} - x^2$ being $= n$, $\overline{a^3-x^3}|^{2}$ will be $= \overline{n+x^2}|^{3}$, and $a^6-2a^3x^3 = n^3+3n^2x^2+3nx^4$: From whence it is evident, that, if n be multiplied by $n^2+3nx^2+3x^4$, the product will be equal to $a^6-2a^3x^3$; and therefore it is plain, that

$$\frac{\overline{a^3-x^3}|^{\frac{2}{3}} - x^2}{\overline{a^3-x^3}|^{\frac{2}{3}}} \text{ is } = \frac{a^6-2a^3x^3}{\overline{a^3-x^3}|^{\frac{2}{3}} \times \overline{n^2+3nx^2+3x^4}}.$$

From theſe few articles (here put by way of notes) may be collected all that is neceſſary to be known, in regard to performing the principal operations in all the propoſitions conſidered in this eſſay.

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